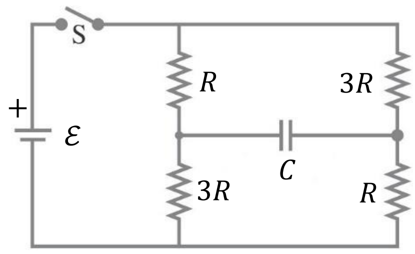
PHYS 102 – General Physics II Midterm Exam 2

Duration: 120 minutes | Saturday, 20 April 2024; 09:30

1. In the circuit shown, the capacitor is initially uncharged, and the switch S is closed at time $t = 0$.



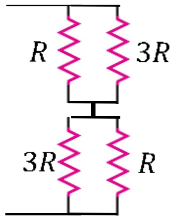
(a) (6 Pts.) Find the current supplied by the battery at $t = 0$.

(b) (6 Pts.) What is the charge on the capacitor when it is fully charged?

(c) (13 Pts.) After the capacitor is fully charged the switch S is opened. How long does it take for the capacitor to discharge until it has half of its initial charge?

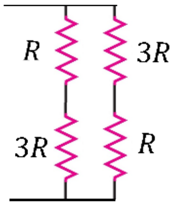
Solution: (a) At $t = 0$, the capacitor is uncharged and hence, there is no potential drop across it. Therefore, it acts as a short circuit. The equivalent resistance is

$$R_{eq} = \frac{3R^2}{R+3R} + \frac{3R^2}{R+3R} = \frac{3}{2}R \quad \rightarrow \quad i = \frac{\mathcal{E}}{R_{eq}} = \frac{2\mathcal{E}}{3R}.$$

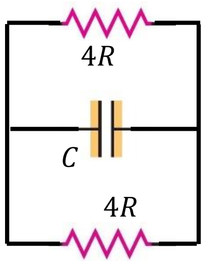


(b) When the capacitor is fully charged no current passes through it. In this case, we have $R_{eq} = 2R$, and the current through each branch is $i_1 = i_2 = \mathcal{E}/4R$. Applying loop rule to the upper loop, we get

$$-V_R - V_C + V_{3R} = 0 \quad \rightarrow \quad -Ri_1 - \frac{Q}{C} + 3R\, i_2 = 0 \quad \rightarrow \quad -\frac{\mathcal{E}}{4} - \frac{Q}{C} + \frac{3\mathcal{E}}{4} = 0 \quad \rightarrow \quad Q = \frac{1}{2}\mathcal{E}C\,.$$



(c) When the switch S is opened, the fully charged capacitor will discharge through the equivalent resistence $R_{eq} = 2R$. Therefore,

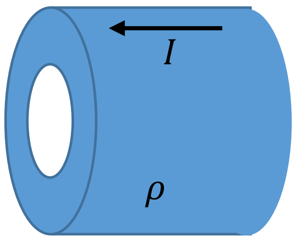


$$q(t) = Q\, e^{-t/R_{eq}C} \quad \rightarrow \quad q(t) = Q\, e^{-t/2RC}\,.$$

At time $t = T$ with the capacitor half charged, we have

$$q(T) = \tfrac{1}{2}Q = Q\, e^{-T/2RC} \quad \rightarrow \quad T = 2RC\ln 2\,.$$

2. A steady current I flows in a very long hollow cylndrical conductor of inner radius R_1 and outer radius R_2. The current is distributed uniformly over the cross section of the conductor whose resistivity is ρ.



(a) (5 Pts.) What is the resistance per unit lenth of the conductor?

(b) (5 Pts.) What is the magnitude and the direction of the electric field in the conductor?

(c) (15 Pts.) What is the magnitude $B(r)$ of the magnetic field inside the conductor for $R_1 < r < R_2$?

Solution: (a)

$$R = \frac{\rho L}{A} \quad \rightarrow \quad \frac{R}{L} = \frac{\rho}{\pi(R_2^2 - R_1^2)}.$$

(b)

$$\vec{\mathbf{E}} = \rho\,\vec{\mathbf{J}} \quad \rightarrow \quad E = \frac{\rho I}{\pi(R_2^2 - R_1^2)} = \frac{IR}{L}.$$

Direction of the electric field is along the axis in the direction of the curent.

(c) Due to the cylindrical symmetry of the current distribution, magnetic field is tangent to circles centered at the symmetry axis, and is constant on each circle. Therefore, choosing a circular Ampèrian loop with radius r centered at the symmetry axis , we have

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{\text{enc}} \quad \rightarrow \quad (2\pi r)B(r) = \mu_0 \frac{I\pi(r^2 - R_1^2)}{\pi(R_2^2 - R_1^2)} \quad \rightarrow \quad B(r) = \frac{\mu_0 I}{2\pi r}\left(\frac{r^2 - R_1^2}{R_2^2 - R_1^2}\right), \qquad R_1 < r < R_2 .$$

3. The magnetic field in a region of space is given as $\vec{\mathbf{B}} = B_z\, \hat{\mathbf{k}}$.

(a) (12 Pts.) Determine the components of the magnetic force on a particle with charge q at the instant the velocity of the particle is $\vec{\mathbf{v}} = v_x\, \hat{\mathbf{i}} + v_z\, \hat{\mathbf{k}}$.

(b) (13 Pts.) If a long wire carrying a current I in the positive y direction is placed on the y axis, what will be the force per unit length on the wire?

Solution: (a)

$$\vec{\mathbf{F}}_B = q\, \vec{\mathbf{v}} \times \vec{\mathbf{B}} \quad \rightarrow \quad \vec{\mathbf{F}}_B = q\left(v_x\, \hat{\mathbf{i}} + v_z\, \hat{\mathbf{k}}\right) \times \left(B_z\, \hat{\mathbf{k}}\right) \quad \rightarrow \quad \vec{\mathbf{F}}_B = q v_x B_z\, \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -q v_x B_z\, \hat{\mathbf{j}}\,.$$

Components are $F_x = 0\,,\ F_y = -q v_x B_z\,,\ F_z = 0\,.$

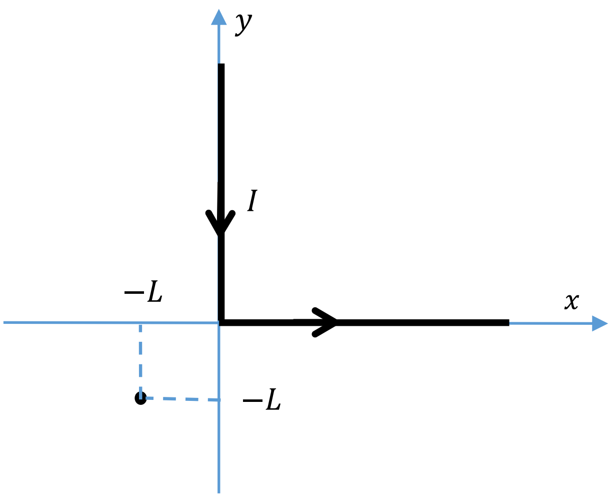
(b)

$$\vec{\mathbf{F}}_B = I\, \vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} \quad \rightarrow \quad \frac{\vec{\mathbf{F}}_B}{\ell} = (I\, \hat{\mathbf{j}}) \times \left(B_z\, \hat{\mathbf{k}}\right) \quad \rightarrow \quad \frac{\vec{\mathbf{F}}_B}{\ell} = I B_z\, \hat{\mathbf{j}} \times \hat{\mathbf{k}} = I B_z\, \hat{\mathbf{i}}\,.$$

4. An infinite wire carrying a current I is bent 90 degrees so that it lies on the positive x and positive y axes as shown in the figure.

(a) (7 Pts.) What is the direction of the magnetic field at the point $x = -L, y = -L$?

(b) (18 Pts.) What is the magnitude of the magnetic field at the same point.



Solution: (a) By the right hand rule we know that both segments create magnetic field in the $(-\hat{\mathbf{k}})$ direction (perpendicular to the page and inwards).

(b) Symmetry of the current with respect to the point $(x = -L, y = -L)$ means magnitude of the magnetic field created by the two segments are equal at that point.

$$d\,\vec{\mathbf{F}}_B = \frac{\mu_0 I}{4\pi} \frac{d\vec{\boldsymbol{\ell}} \times (\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3}$$

For the vertical section on the y axis we have $d\vec{\boldsymbol{\ell}} = -dy'\,\hat{\mathbf{j}}$, $\vec{\mathbf{r}} = -L\,\hat{\mathbf{i}} - L\,\hat{\mathbf{j}}$, $\vec{\mathbf{r}}' = y'\hat{\mathbf{j}}$. Therefore

$$\hat{\mathbf{r}} - \hat{\mathbf{r}}' = -L\,\hat{\mathbf{i}} - (L + y')\hat{\mathbf{j}}\,, \qquad |\hat{\mathbf{r}} - \hat{\mathbf{r}}'| = \sqrt{L^2 + (L + y')^2}\,,$$

$$d\,\vec{\mathbf{F}}_B = \frac{\mu_0 I}{4\pi} \frac{(-dy'\,\hat{\mathbf{j}}) \times [-L\,\hat{\mathbf{i}} - (L + y')\hat{\mathbf{j}}]}{[L^2 + (L + y')^2]^{3/2}} \quad \rightarrow \quad d\,\vec{\mathbf{F}}_B = \frac{\mu_0 I}{4\pi} \frac{L dy'(-\,\hat{\mathbf{k}})}{[L^2 + (L + y')^2]^{3/2}}$$

$$\vec{\mathbf{F}}_B = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{L dy'(-\,\hat{\mathbf{k}})}{[L^2 + (L + y')^2]^{3/2}} \quad \rightarrow \quad \vec{\mathbf{F}}_B = \frac{\mu_0 I L}{4\pi} (-\,\hat{\mathbf{k}}) \int_L^\infty \frac{du}{[L^2 + u^2]^{3/2}}$$

The integral can be evaluated by using the substitution $u = L\tan\theta$, $du = L\sec^2\theta\ d\theta$

$$\int \frac{du}{[L^2 + u^2]^{3/2}} = \frac{1}{L^2} \int \cos\theta\ d\theta = \frac{1}{L^2} \sin\theta.$$

Since

$$\tan\theta = \frac{u}{L} \quad \rightarrow \quad \sin\theta = \frac{u}{\sqrt{u^2 + L^2}},$$

$$\int_L^\infty \frac{du}{[L^2 + u^2]^{3/2}} = \frac{1}{L^2} \frac{u}{\sqrt{u^2 + L^2}}\bigg|_L^\infty = \frac{1}{L^2}\left(1 - \frac{1}{\sqrt{2}}\right) \quad \rightarrow \quad \vec{\mathbf{F}}_B = \frac{\mu_0 I}{4\pi L}\left(1 - \frac{1}{\sqrt{2}}\right)(-\,\hat{\mathbf{k}}).$$

For both sections, the result is

$$\vec{\mathbf{F}}_B = \frac{\mu_0 I}{2\pi L}\left(1 - \frac{1}{\sqrt{2}}\right)(-\,\hat{\mathbf{k}}).$$